Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337

TMA 03 2020J

Covers Book C **Cut-off date 31 March 2021**

You can submit this TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on it, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' area of the M337 website.

Your work should be written in good mathematical style, as demonstrated by the example and exercise solutions in the study units. You should explain your solutions carefully, using appropriate notation and terminology, and write in sentences. As usual, you should simplify algebraic answers where possible.

In the wording of the questions:

- *write down* or *state* means 'write down without justification'
- *find*, *determine*, *calculate*, *explain*, *derive*, *evaluate* or *solve* means that we require you to show all your working in giving an answer
- *prove*, *show* or *deduce* means that you should carefully justify each step of your solution.

Make sure to reference any significant result from the module materials that you use, and check that all the conditions of the result are satisfied.

Question 1 (Unit C1) – 33 marks

(a) Let

$$f(z) = \frac{z^2+1}{(2z^2+5z+2)^2}.$$

(i) Show that f has a pole of order 2 at the point $-1/2$, and evaluate the residue of f at $-1/2$. *[4]*

(ii) Use the strategy for evaluating real trigonometric integrals and the result from part (a)(i) to deduce that

$$\int_0^{2\pi} \frac{\cos t}{(4\cos t+5)^2}\,dt = -\frac{8\pi}{27}.$$ *[5]*

(b) Use a method for summing series of even functions to prove that

$$\sum_{n=1}^{\infty} \frac{8}{16n^4-1} = 4 - \pi\coth\frac{\pi}{2}.$$ *[13]*

(c) Use Theorem 5.3 on page 72 of Book C to prove that

$$\int_0^{\infty} \frac{t^{1/2}}{t^3+t}\,dt = \frac{\pi}{\sqrt{2}}.$$ *[11]*

Question 2 (Unit C2) – 34 marks

(a) Sketch the path

$$\Gamma : \gamma(t) = \begin{cases} -1+e^{it} & (t \in [0,2\pi]) \\ 1-e^{-it} & (t \in [2\pi,4\pi]), \end{cases}$$

indicating the directions of increasing values of t.

Mark the points 1 and -1 on your diagram, and write down the winding number of Γ around each of these two points. *[4]*

(b) Determine the number of zeros of the function

$$f(z) = z^5 + iz^3 - 5z^2 + 2$$

in each of the following sets.

(i) $S_1 = \{z : |z| \le 1\}$ *[5]*

(ii) $S_2 = \{z : 1 < |z| \le 2\}$ *[5]*

(iii) $S_3 = \{z : |z| > 2\}$ *[2]*

(c) Let $f(z) = (z-\pi)\sin z$. Find an integer n such that f is n-to-one near the point π. *[3]*

(d) Determine

$$\max\{|\exp(1/z^3)| : 1 \le |z| \le 2\},$$

and find all points at which the maximum is attained, giving your answers in Cartesian form. *[8]*

(e) Consider the series

$$\sum_{n=1}^{\infty} \frac{z^{2n}}{4^n + n^4}.$$

(i) Prove that the series is uniformly convergent on $\{z : |z| \leq r\}$, for $0 < r < 2$. *[4]*

(ii) Prove that the series defines a function that is analytic on $\{z : |z| < 2\}$. *[3]*

Question 3 (Unit C3) – 33 marks

(a) Find a Möbius transformation of the form

$$f(z) = \frac{z + b}{cz + 1},$$

for suitable real numbers b and c, that has fixed points $3i$ and $-3i$. *[3]*

(b) Let f be the Möbius transformation

$$f(z) = \frac{iz + 7}{z - 2},$$

and let $C = \{z : |z - 1| = 2\}$.

(i) Find an equation for the image circle $f(C)$ in Apollonian form. *[6]*

(ii) Determine the centre and radius of $f(C)$. *[2]*

(c) Let

$$\mathcal{R} = \{z : |z| < 1,\ \operatorname{Im} z - \operatorname{Re} z > 1\},$$
$$\mathcal{S} = \{z : 0 < \operatorname{Arg} z < \pi/4\},$$
$$\mathcal{T} = \{z : \operatorname{Re} z < 0,\ \operatorname{Im} z > 0\}.$$

(i) Sketch the regions $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$. *[3]*

(ii) Write down a positive number λ such that the point $\lambda(-1 + i)$ lies on the boundary of $\mathcal{R}$. *[1]*

(iii) Determine a one-to-one conformal mapping f from $\mathcal{R}$ onto $\mathcal{S}$. *[7]*

(iv) Determine a one-to-one conformal mapping g from $\mathcal{S}$ onto $\mathcal{T}$. *[4]*

(v) Hence determine a one-to-one conformal mapping h from $\mathcal{R}$ onto $\mathcal{T}$. *[2]*

(vi) Find the rule for the inverse function h^{-1} of the function h found in part (c)(v). *[3]*

(vii) Determine the image under h of the set $\widehat{\mathbb{C}} - \mathcal{R}$. *[2]*